VECTORS OF SMALLEST SLOPE

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Background

Translation Surfaces : A 3D surface obtained through “gluing” the sides of a 2D polygon together.

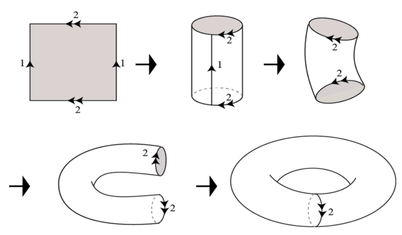


Fig. 1: How an unit square becomes a torus.

Matrix Transformations : Matrices that can act upon translation surfaces to change them.

Fig. 2: How an unit square becomes a torus.

Two surfaces are in the same strata if you can cut and paste parts to make them the same.

Fig. 3: Surfaces in the same strata.

Genus : The number of holes in a translation surface.

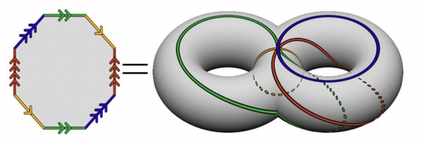


Fig. 4: Genus-2.

Cone Points : Cone points are areas of extra angle caused by trying to “glue” a polygon with over 2π radians of angle.

Fig. 5: An image of a square and an octagon.

Surfaces with the same number of cone points, same excess angle at cone points, and same genus are in the same strata.

Saddle Connections : Straight lines connecting cone points that do not intersect any other cone points. Saddle connections are an ordered list of vectors.

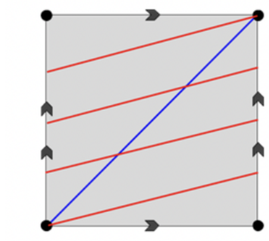


Fig. 6: Saddle connections on an unit square.

Methods

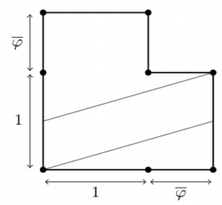
Horocycle Flow: We use a matrix, h_s, to analyze saddle connections.

$$h_s = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$$

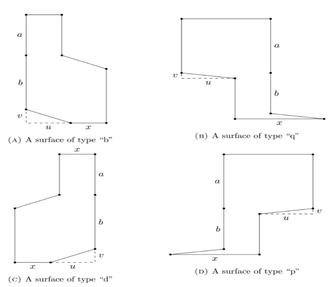
When s is the vector slope, the vector becomes horizontal. When the same s is used, slope differences are preserved.

Transversal : The set of all translation surface with a horizontal saddle connection of length less than or equal to one.

Golden L : Translation surface obtained from an L-shaped polygon by gluing opposite sides by horizontal and vertical translations.



Four surfaces come as a result of cutting pasting parts of the Golden L. Using the variables of *a,b,v,u,x*, we try to determine translation surfaces and which shapes among B, Q, D, P is referenced to.



(a) A surface of type “b”

(b) A surface of type “q”

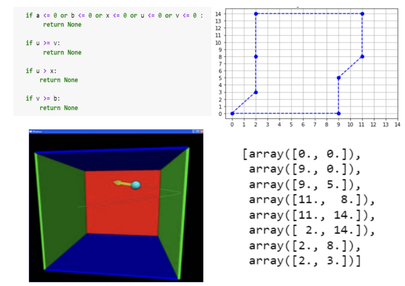
(c) A surface of type “d”

(d) A surface of type “p”

Coding

There are two coding approaches we are taking:

1. A visual 3D “Ball In A Box” approach.
   - Shoot an elements from many different angles on each saddle connection
   - Trace the path of elements across a translation surface making sure to loop around sides when applicable
   - When an element strikes another (or the same) saddle connection, record the slope of its path
   - Once all slopes are gathered, check them all for a horizontal component that is less than or equal to one, and find the smallest slope
   - This allows us to do analysis on many types of translation surfaces, so we are not just limited to the four we are studying
2. An interactive slider in which the user can make new shapes from B, Q, D, P.
   - We start by asking the user to explicitly define what a, b, x, v, and u are
   - The user must choose what type of shape they want
   - A list of all possible smallest slopes have been calculated by hand for each type of surface depending on what a, b, x, v, and u are.
   - Figure out a list of slopes from these equations, and analyze them to find the smallest slope of horizontal component of less than or equal to one
   - Put the surface under horocycle flow according to this found slope
   - Paste the new translated surface back into one of the four main types and perform the calculations again
   - We have a early stage interactive demo for how this could work



```
if a <= 0 or b <= 0 or x <= 0 or u <= 0 or v <= 0 :
    return None

if u >= v:
    return None

if u > x:
    return None

if v >= b:
    return None
```

```
[array([0., 0.]),
 array([9., 0.]),
 array([9., 5.]),
 array([11.,  8.]),
 array([11., 14.]),
 array([ 2., 14.]),
 array([2., 8.]),
 array([2., 3.])]
```

References

[1] Jayadev Athreya, Jon Chaika, and Samuel Lelièvre. “The gap distribution of slopes on the golden L”. In: (Aug. 2013). DOI: 10.1090/conm/631/12595.

[2] Jayadev S. Athreya and Yitwah Cheung. “A Poincaré Section for the Horocycle Flow on the Space of Lattices”. In: *International Mathematics Research Notices* 2014.10 (Jan. 2013). _eprint: https://academic.oup.com/imrn/article-pdf/2014/10/2643/18853800/rnt003.pdf, pp. 2643–2690. ISSN: 1073-7928. DOI: 10.1093/imrn/rnt003. URL: https://doi.org/10.1093/imrn/rnt003.

[3] Caglar Uyanik and Grace Work. “The Distribution of Gaps for Saddle Connections on the Octagon”. In: *International Mathematics Research Notices* 2016 (Oct. 2016). DOI: 10.1093/imrn/rnv317.

[4] Grace Work. *A Transversal for horocycle flow on H(2)*. 2016. DOI: 10.48550/ARXIV.1607.05777. URL: https://arxiv.org/abs/1607.05777.